

Stationarity of multivariate particle systems ^{*}

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January 24, 2012

Abstract

A particle system is a family of i.i.d. stochastic processes with values translated by Poisson points. We obtain conditions that ensure the stationarity in time of the particle system in \mathbb{R}^d and in some cases provide a full characterisation of the stationarity property. It is shown how the characterisation problem relates to solutions of two-sided convolution equations.

1 Introduction

A Poisson process in the Euclidean space \mathbb{R}^d is stationary if its intensity measure is proportional to the Lebesgue measure. More general Poisson processes can be defined on richer spaces, e.g. the space of functions or sets. While in these cases often there is no analogue of the Lebesgue measure, invariance properties of the process can be defined with respect to transformations that account for the internal structure of the relevant phase space.

One of most spectacular examples of this situation is due to Kabluchko [5], who considered the following situation. Let Π be a Poisson point process in \mathbb{R} and let $\{\xi_i, i \geq 1\}$ be i.i.d. copies of a real-valued stochastic process $\xi(t)$, $t \in \mathbb{R}^m$. Define the family of functions $x_i + \xi_i(t)$, $t \in \mathbb{R}^m$, for $x_i \in \Pi$, which (under appropriate integrability conditions on the intensity of the Poisson process) becomes a point process in the space of functions on \mathbb{R}^m .

^{*}Supported by Swiss National Science Foundation Project Nr. 200021-137527. The authors are grateful to Zakhar Kabluchko for helpful discussions at the earlier stage of this work.

For any $t \in \mathbb{R}^m$, $N(t) = \{x_i + \xi_i(t) : i \geq 1\}$ is the Poisson point process in \mathbb{R} . Sometimes, the point process $N(t)$ formed by the values of the translated function is stationary in time even if ξ is not stationary. It is important not to mix this concept with stationarity in \mathbb{R} , where the points lie.

Kabluchko [5] characterised the cases when a real-valued Gaussian process ξ gives rise to a stationary point system $N(t)$ called a stationary *Gaussian system* assuming that the intensity measure Λ of Π satisfies $\int_{\mathbb{R}} e^{-\varepsilon x^2} \Lambda(dx) < \infty$ for all $\varepsilon > 0$. The stationary Gaussian systems are given by the following three classes.

- (i) Λ is an arbitrary measure on \mathbb{R} and ξ is a stationary Gaussian process.
- (ii) Λ is proportional to the Lebesgue measure on \mathbb{R} and $\xi(t) = W(t) + b(t) + c$, where W is a centred Gaussian process with stationary increments, $b(t)$ is an *additive function*, i.e. $b(t+s) = b(t) + b(s)$ for all t and s , and $c \in \mathbb{R}$ is a constant.
- (iii) The density of Λ is proportional to $e^{-\lambda x}$, $x \in \mathbb{R}$, with $\lambda \neq 0$, and $\xi(t) = W(t) - \lambda \sigma^2(t)/2 + c$, where W is a centred Gaussian process with stationary increments and variance $\sigma^2(t)$, and $c \in \mathbb{R}$ is a constant.

The aim of this paper is to provide a partial generalisation of the above result for the case when ξ takes values in a higher-dimensional Euclidean space, which is also mentioned in [5] as an interesting open problem. We show the relationship of the full characterisation problem to yet unsolved problems in real analysis.

2 Multivariate particle systems

Let $\{\xi_i, i \geq 1\}$ be i.i.d. copies of a \mathbb{R}^d -valued stochastic process $\xi(t)$, $t \in \mathbb{R}$. All subsequent results can be easily generalised and remain valid for processes ξ with argument t from a higher-dimensional Euclidean space.

Furthermore, let $\Pi = \{x_i, i \geq 1\}$ be a Poisson point process in \mathbb{R}^d independent of the ξ_1, ξ_2, \dots . We call the process

$$N(t) = \{x_i + \xi_i(t), i \geq 1\}, \quad t \in \mathbb{R},$$

a *particle system*, so that a particle system is a stochastic process with values in the space of point configurations (or counting measures). Since the distribution of Π is completely determined by its intensity measure Λ , we say that the particle system (Λ, ξ) is generated by measure Λ and process ξ .

By the finite-dimensional distributions of N we mean the distribution of the point process in \mathbb{R}^{dn} given by

$$N(t_1, \dots, t_n) = \{(x_i + \xi_i(t_1), \dots, x_i + \xi_i(t_n)), i \geq 1\}, \quad t_1, \dots, t_n \in \mathbb{R}.$$

Denote by P_{t_1, \dots, t_n} the finite-dimensional distributions of ξ , in particular P_t is the distribution of $\xi(t)$. From now on we will always assume that the convolution $\Lambda * P_t$ is a locally finite measure for all $t \in \mathbb{R}$. The following result is easy to obtain, e.g. using the probability generating functional of a Poisson process, see [1].

Proposition 2.1. *If $\Lambda * P_t$ is a locally finite measure for all $t \in \mathbb{R}$, then, for all $t_1, \dots, t_n \in \mathbb{R}$, $N(t_1, \dots, t_n)$ is Poisson point process in \mathbb{R}^{dn} with locally finite intensity measure*

$$\Lambda_{t_1, \dots, t_n}(A) = \int_{\mathbb{R}^d} P_{t_1, \dots, t_n}(A - x) \Lambda(dx) \quad (2.1)$$

for all Borel $A \subset \mathbb{R}^{dn}$, where $A - x$ is A translated by (x, \dots, x) composed of n copies of $x \in \mathbb{R}^d$.

The main question addressed throughout this paper is to characterise all pairs (Λ, ξ) , such that the corresponding particle system N is stationary. Since the distribution of a Poisson point process is determined by its intensity measure, we immediately obtain the following result.

Proposition 2.2. *The particle system generated by Λ and ξ is stationary if and only if*

$$\Lambda_{t_1, \dots, t_n} = \Lambda_{t_1+s, \dots, t_n+s} \quad (2.2)$$

for all $s, t_1, \dots, t_n \in \mathbb{R}$.

3 Convolution equations

The stationarity condition (2.2) is in fact a system of convolution equations of the form

$$P_{t_1, \dots, t_n} * \tilde{\Lambda} = P_{t_1+s, \dots, t_n+s} * \tilde{\Lambda}, \quad (3.1)$$

where $\tilde{\Lambda}$ is the measure obtained by uplifting Λ to the diagonal in \mathbb{R}^{dn} . In general notation, these equations are of the type

$$\sigma_1 * \mu = \sigma_2 * \mu, \quad (3.2)$$

where σ_1 and σ_2 are probability measures and μ is an unknown locally finite measure. If σ_2 can be decomposed as $\sigma_2 = \sigma_1 * \sigma$ (or if $\sigma_1 = \sigma_2 * \sigma$), then (3.2) simplifies to

$$\mu = \mu * \sigma \quad (3.3)$$

for another measure μ . This convolution equation was solved by Dény [2]. Namely, if the support of σ is the whole \mathbb{R}^d , then all solutions of (3.3) are mixtures of exponential measures. i.e.

$$\mu = \int_E \mathfrak{e}_\lambda Q(d\lambda), \quad (3.4)$$

where \mathfrak{e}_λ is the measure on \mathbb{R}^d with density $e^{-\langle \lambda, x \rangle}$, $x \in \mathbb{R}^d$, and Q is a measure on the set $E = E_\sigma$ with

$$E_\sigma = \left\{ \lambda \in \mathbb{R}^d : \int_{\mathbb{R}^d} e^{\langle \lambda, x \rangle} \sigma(dx) = 1 \right\}. \quad (3.5)$$

In particular, if ξ is a real-valued Gaussian process with non-constant variance $\sigma^2(t)$, $t \in \mathbb{R}$, then there exist $t_1, t_2 \in \mathbb{R}$ such that $\sigma^2(t_2) > \sigma^2(t_1)$, so that the first convolution equation $P_{t_1} * \Lambda = P_{t_2} * \Lambda$ can be reduced to the Dény convolution equation (3.3) with σ being the normal law of the variance $\sigma^2(t_2) - \sigma^2(t_1)$. Hence $\Lambda * P_t$ is necessarily a mixture of exponential measures, which is the crucial argument in the characterisation of stationary Gaussian systems in [5].

In the multivariate case the above argument does not apply any longer, since the difference of the covariance matrices of $\xi(t_2)$ and $\xi(t_1)$ may be neither positive nor negative definite. In the spirit of (3.5), define

$$E_{\sigma_1 \sigma_2} = \left\{ \lambda \in \mathbb{R}^d : \int_{\mathbb{R}^d} e^{\langle \lambda, x \rangle} \sigma_1(dx) = \int_{\mathbb{R}^d} e^{\langle \lambda, x \rangle} \sigma_2(dx) \right\}. \quad (3.6)$$

While each measure μ given by (3.4) with $E = E_{\sigma_1 \sigma_2}$ satisfies the convolution equation (3.2), there exist solutions of (3.2) not in the form (3.4).

Example 3.1. Let σ_1 and σ_2 be bivariate centred normal distributions with covariance matrices

$$\Sigma_1 = \begin{pmatrix} 1 + c_1^2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \Sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 + c_2^2 \end{pmatrix}.$$

for some constants $c_1, c_2 > 0$. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $\int_{\mathbb{R}} g(x + y) e^{-y^2/2} dy$ is finite for all $x \in \mathbb{R}$. Then each measure μ_g with the

density $g(c_1^{-1}x_1 + c_2^{-1}x_2)$ satisfies (3.2). Indeed, substitution $z = c_1^{-1}c_2y$ yields that

$$\frac{1}{c_1} \int_{\mathbb{R}} g\left(\frac{x_1}{c_1} + \frac{x_2}{c_2} - \frac{y}{c_1}\right) e^{-\frac{1}{2}\left(\frac{y}{c_1}\right)^2} dy = \frac{1}{c_2} \int_{\mathbb{R}} g\left(\frac{x_1}{c_1} + \frac{x_2}{c_2} - \frac{z}{c_2}\right) e^{-\frac{1}{2}\left(\frac{z}{c_2}\right)^2} dz.$$

It remains to note that the both sides of this equality are up to the same constant the densities of the convolution of μ_g and the centred normal distributions with covariance matrices

$$\begin{pmatrix} c_1^2 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & c_2^2 \end{pmatrix}$$

Note that $E_{\sigma_1\sigma_2} = \{(\lambda_1, \lambda_2) : c_1|\lambda_1| = c_2|\lambda_2|\}$. For instance, if $g(x) = x^2$ then there exists no measure Q such that $\mu_g = \int_{E_{\sigma_1\sigma_2}} \mathbf{e}_\lambda Q(d\lambda)$.

Unfortunately there is no general result describing solutions of (3.2). The two-sided convolution equation can be written as $\mu * \nu = 0$ for a signed measure ν with finite total variation. If ν has a bounded support, then the density of μ solving this equation is called mean periodic function. Typical examples of mean periodic functions are exponential polynomials, i.e. sums of products of polynomials and exponential functions. While exponential polynomials are dense in the family of mean periodic functions on the line [3], this is unknown for higher-dimensional spaces. The situation with ν having unbounded support (like in case of Gaussian measures) is even less explored. Thus, there are no analytical tools suitable to identify solutions of the two-sided convolution equation (3.2).

4 Multivariate stationarity

In this section we characterise the stationarity conditions for some (but still rather general) intensity measures Λ .

4.1 Exponential measures.

Consider candidates for the solutions of (3.1) of the form $\Lambda = \mathbf{e}_\lambda$ for $\lambda \in \mathbb{R}^d$. It is easy to see that necessarily $\lambda \in E_{P_t P_s}$ (see (3.6)) for any $t, s \in \mathbb{R}$. The convolution $\mathbf{e}_\lambda * P_t$ is locally finite if and only if

$$\mathbf{E} e^{\langle \lambda, \xi(t) \rangle} < \infty \quad \text{for all } t \in \mathbb{R}. \quad (4.1)$$

Then the characteristic function with a complex argument in its first coordinate

$$\begin{aligned}\varphi_{t_1, \dots, t_n}(u_1 - \imath\lambda, \dots, u_n) \\ = \mathbf{E} \exp\{\imath(\langle u_1 - \imath\lambda, \xi(t_1) \rangle + \langle u_2, \xi(t_2) \rangle + \dots + \langle u_n, \xi(t_n) \rangle)\}\end{aligned}$$

exists for all $u_1, \dots, u_n \in \mathbb{R}^d$, where \imath is the imaginary unit.

Theorem 4.1. *Assume that (4.1) holds. The particle system generated by \mathbf{e}_λ and ξ is stationary if and only if*

$$\varphi_{t_1, \dots, t_n}(u_1 - \imath\lambda, \dots, u_n) = \varphi_{t_1+s, \dots, t_n+s}(u_1 - \imath\lambda, \dots, u_n) \quad (4.2)$$

for all $n \geq 1$, $s, t_1, \dots, t_n \in \mathbb{R}$ and $u_1, \dots, u_n \in \mathbb{R}^d$ satisfying $\sum_{i=1}^n u_i = 0$.

Proof. The proof follows the idea of [6, Prop. 6]. Let A be a bounded Borel set in \mathbb{R}^{dn} . Then

$$\begin{aligned}\Lambda_{t_1, \dots, t_n}(A) &= \mathbf{E} \int_{\mathbb{R}^d} \mathbf{1}_{(\xi(t_1)+x, \dots, \xi(t_n)+x) \in A} e^{-\langle \lambda, x \rangle} dx \\ &= \int_{\mathbb{R}^d} \mu_{t_1, \dots, t_n}(A - z) e^{-\langle \lambda, z \rangle} dz, \quad (4.3)\end{aligned}$$

where μ is a measure on \mathbb{R}^{dn} given by

$$\mu_{t_1, \dots, t_n}(A) = \mathbf{E} \left[\mathbf{1}_{(0, \xi(t_2) - \xi(t_1), \dots, \xi(t_n) - \xi(t_1)) \in A} e^{\langle \lambda, \xi(t_1) \rangle} \right]. \quad (4.4)$$

Since μ_{t_1, \dots, t_n} is supported by the subspace $\{(x_1, \dots, x_n) \in \mathbb{R}^{dn} : x_1 = 0\}$, the decomposition $\Lambda_{t_1, \dots, t_n}(A) = \int \mu_{t_1, \dots, t_n}(A - z) e^{-\langle \lambda, z \rangle} dz$ is unique, e.g. see [7, Th. 15.3.3]. Finally, note that the Fourier transform of μ_{t_1, \dots, t_n} is given by

$$\hat{\mu}_{t_1, \dots, t_n}(u_1, \dots, u_n) = \varphi_{t_1, \dots, t_n} \left(-\imath\lambda - \sum_{i=2}^n u_i, u_2, \dots, u_n \right).$$

□

A similar proof with the Laplace transform instead of the Fourier transform yields the following result.

Proposition 4.2. *Assume that the Laplace transform*

$$\psi_{t_1, \dots, t_n}(u_1, \dots, u_n) = \mathbf{E} \exp\{\langle u_1, \xi(t_1) \rangle + \dots + \langle u_n, \xi(t_n) \rangle\} \quad (4.5)$$

exists for all $u_1, \dots, u_n \in \mathbb{R}^d$ such that $\sum_{i=1}^n u_i = \lambda$. Then the particle system $(\mathbf{e}_\lambda, \xi)$ is stationary if and only if $\psi_{t_1, \dots, t_n}(u_1, \dots, u_n) = \psi_{t_1+s, \dots, t_n+s}(u_1, \dots, u_n)$ for all $n \geq 1$, $s, t_1, \dots, t_n \in \mathbb{R}$ and $u_1, \dots, u_n \in \mathbb{R}^d$ satisfying $\sum_{i=1}^n u_i = \lambda$.

For Gaussian processes we can give a more precise statement. Denote by $\Sigma(t_1, t_2)$ the covariance matrix of $\xi(t_1)$ and $\xi(t_2)$, in particular $\Sigma(t, t)$ is the covariance matrix of $\xi(t)$. It is important to note that, unlike the univariate case, $\Sigma(t_1, t_2)$ may differ from $\Sigma(t_2, t_1)$, namely $\Sigma(t_2, t_1) = \Sigma(t_1, t_2)^\top$. Furthermore, the covariance matrix (variogram) of $\xi(t_2) - \xi(t_1)$ is given by

$$\Gamma(t_1, t_2) = \Sigma(t_2, t_2) - \Sigma(t_1, t_2) - \Sigma(t_2, t_1) + \Sigma(t_1, t_1).$$

We say that multivariate Gaussian process ξ has *wide sense stationary increments* if and only if $\Gamma(t_1, t_2)$ depends only on the difference $t_1 - t_2$. In the univariate case, this property is equivalent to the fact that $\xi(t + s) - \xi(t)$, $t \in \mathbb{R}$, is stationary for each $s \in \mathbb{R}$, see [5, Lemma 1], while in the multivariate case this is not so.

Example 4.3. Let $\xi^1(t) = W(t)$ and let $\xi^2(t) = W(t + h)$ for some fixed h , where W is the Wiener process. Then $\mathbf{E}\xi^1(t_1)\xi^2(t_2)$ is not necessarily equal to $\mathbf{E}\xi^1(t_2)\xi^2(t_1)$, so that $\Sigma(t_1, t_2)$ is not necessarily symmetric.

Theorem 4.4. *The measure \mathbf{e}_λ together with a Gaussian process ξ generate a stationary particle system if and only if*

$$\xi(t) = W(t) - \frac{1}{2}\Sigma(t, t)\lambda + b(t) + c, \quad t \in \mathbb{R}, \quad (4.6)$$

where W is a centred Gaussian process with wide sense stationary increments and variance $\Sigma(t, t)$, $c \in \mathbb{R}^d$ is deterministic, and $b : \mathbb{R} \rightarrow \mathbb{R}^d$ is a function orthogonal to λ such that

$$b(t_2) - b(t_1) + \frac{1}{2}(\Sigma(t_2, t_1) - \Sigma(t_1, t_2))\lambda \quad (4.7)$$

depends only on the difference $t_2 - t_1$.

Remark 4.5. If $\lambda = 0$, condition (4.7) implies that $b(t) - b(0)$ is an additive function, see [5, Lemma 2]. This is also the case if $\Sigma(t_1, t_2)$ is symmetric for all t_1 and t_2 , e.g. in the univariate case where the orthogonality of b and λ implies that b vanishes if $\lambda \neq 0$.

We use the following lemma, that is easy to prove by a direct computation.

Lemma 4.6. *Consider all Gaussian vectors in the Euclidean space \mathbb{R}^n whose Laplace transform $\psi(u)$ is given for all u from $\mathbb{L} + a$, where \mathbb{L} is a linear subspace of \mathbb{R}^n and $a \in \mathbb{R}^n$. Then all these vectors share the same values of $A^\top \Sigma A$, $A^\top(m + \Sigma a)$ and $\langle m, a \rangle + \frac{1}{2}\langle a, \Sigma a \rangle$, where m and Σ are the mean and covariance matrix of the corresponding vector and A denotes any projection of \mathbb{R}^n onto \mathbb{L} .*

Proof of Theorem 4.4. The sufficiency follows by the explicit writing of the Laplace transform of $(\xi(t_1), \dots, \xi(t_n))$. For the necessity, let $n = 2$ and apply Lemma 4.6 and Proposition 4.2 with $\mathbb{L} = \{(u_1, u_2) \in \mathbb{R}^{2d} : u_1 + u_2 = 0\}$ and $a = (\lambda, 0) \in \mathbb{R}^{2d}$. Define block matrices

$$A = \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma(t_1, t_1) & \Sigma(t_1, t_2) \\ \Sigma(t_2, t_1) & \Sigma(t_2, t_2) \end{pmatrix},$$

where I is the d -dimensional unit matrix. Note that A defines a projection on \mathbb{L} and Σ is the covariance of $(\xi(t_1), \xi(t_2))$. Then all elements of $A^\top \Sigma A$ are proportional to $\Gamma(t_1, t_2)$, meaning that $W(t) = \xi(t) - \mathbf{E}\xi(t)$, $t \in \mathbb{R}$, has wide sense stationary increments.

Define $m(t) = \mathbf{E}\xi(t)$. Calculating $A^\top(m + \Sigma a)$ with $m = (m(t_1), m(t_2))$ it is easy to see that

$$m(t_2) - m(t_1) + (\Sigma(t_1, t_1) - \Sigma(t_2, t_1))\lambda \quad (4.8)$$

is invariant after (t_1, t_2) is replaced by $(t_1 + s, t_2 + s)$. Denoting

$$b(t) = m(t) + \frac{1}{2}\Sigma(t, t)\lambda$$

and using the fact that $\Gamma(t_1, t_2) = \Gamma(t_1 + s, t_2 + s)$, we arrive at (4.7). Furthermore,

$$\langle m, a \rangle + \frac{1}{2}\langle a, \Sigma a \rangle = \langle m(t_1), \lambda \rangle + \frac{1}{2}\langle \lambda, \Sigma(t_1, t_1)\lambda \rangle = \langle b(t), \lambda \rangle$$

does not depend on t_1 , so that $\langle b(t), \lambda \rangle$ is constant. Finally, set $c = b(0)$ and replace $b(t)$ by $b(t) - b(0)$. \square

4.2 Mixtures of exponential measures.

Assume that $\Lambda = \int_E \mathbf{e}_\lambda Q(d\lambda)$, where Q is a measure supported by $E \subset \mathbb{R}^d$, so that Λ is locally finite.

Theorem 4.7. *Assume that (4.1) holds for all λ from an open neighbourhood U of E . The particle system generated by $\Lambda = \int \mathbf{e}_\lambda Q(d\lambda)$ and ξ is stationary if and only if the system $(\mathbf{e}_\lambda, \xi)$ is stationary for all $\lambda \in E$.*

Proof. We only need to prove the necessity. For $v \in \mathbb{R}^d$ define

$$E_1 = \{\lambda \in E : \langle \lambda, v \rangle < 1\}, \\ E_2 = \{\lambda \in E : \langle \lambda, v \rangle \geq 1\}.$$

Let $\Lambda_i = \int_{E_i} \mathbf{e}_\lambda Q(d\lambda)$, $i = 1, 2$. Without loss of generality assume that neither Λ_1 nor Λ_2 is the zero measure. Let A be a bounded Borel set. Since Λ satisfies (2.2),

$$\Lambda_1 * (P_{t_1, \dots, t_n} - P_{t_1+s, \dots, t_n+s})(A) = \Lambda_2 * (P_{t_1+s, \dots, t_n+s} - P_{t_1, \dots, t_n})(A). \quad (4.9)$$

Assume that (4.9) is positive. Since $Q(E_1) > 0$,

$$\begin{aligned} \Lambda_1 * (P_{t_1, \dots, t_n} - P_{t_1+s, \dots, t_n+s})(A+v) &> e^{-1} \Lambda_1 * (P_{t_1, \dots, t_n} - P_{t_1+s, \dots, t_n+s})(A), \\ \Lambda_2 * (P_{t_1+s, \dots, t_n+s} - P_{t_1, \dots, t_n})(A+v) &\leq e^{-1} \Lambda_2 * (P_{t_1+s, \dots, t_n+s} - P_{t_1, \dots, t_n})(A). \end{aligned}$$

In view of (4.9),

$$\Lambda_1 * (P_{t_1, \dots, t_n} - P_{t_1+s, \dots, t_n+s})(A+v) > \Lambda_2 * (P_{t_1+s, \dots, t_n+s} - P_{t_1, \dots, t_n})(A+v).$$

Rearranging the terms yields that

$$(\Lambda_1 + \Lambda_2) * P_{t_1, \dots, t_n}(A+v) > (\Lambda_1 + \Lambda_2) * P_{t_1+s, \dots, t_n+s}(A+v),$$

which contradicts that $\Lambda = \Lambda_1 + \Lambda_2$ satisfies (2.2). A similar argument excludes the negativity of (4.9), and therefore Λ_1 and Λ_2 satisfy (2.2) for all bounded Borel A .

Consider any $\lambda_0 \in E$. By cutting E with hyperplanes, it is possible to construct a sequence of relatively compact sets $E_k \subset E$, $k \geq 1$, such that $E_k \downarrow \{\lambda_0\}$, the closure of E_1 is a subset of U and $\Lambda_k = \int_{E_k} \mathbf{e}_\lambda Q(d\lambda)$ satisfies (2.2) for all k . Since (2.2) is scale invariant, it also holds for $\tilde{\Lambda}_k = \int_{E_k} \mathbf{e}_\lambda \tilde{Q}_k(d\lambda)$ with $\tilde{Q}_k(\cdot) = Q(\cdot)/Q(E_k)$. For all k ,

$$\inf_{\lambda \in E_k} e^{-\langle \lambda, x \rangle} \leq \int_{E_k} e^{-\langle \lambda, x \rangle} \tilde{Q}_k(d\lambda) \leq \sup_{\lambda \in E_k} e^{-\langle \lambda, x \rangle}, \quad (4.10)$$

Since the both sides of (4.10) converge to $e^{-\langle \lambda_0, x \rangle}$, $\tilde{\Lambda}_k(A) \rightarrow \mathbf{e}_{\lambda_0}(A)$ for all measurable A .

It remains to show that the limiting measure satisfies (2.2). By (4.3),

$$\tilde{\Lambda}_k * P_{t_1, \dots, t_n}(A) = \int_{\mathbb{R}^d} \int_{E_k} \mu_{t_1, \dots, t_n}(A-x) e^{-\langle \lambda, x \rangle} \tilde{Q}_k(d\lambda) dx, \quad (4.11)$$

where $\mu_{t_1, \dots, t_n}(A)$ is defined in (4.4).

Since $\tilde{Q}_k(E_k) = 1$ and $\mu_{t_1, \dots, t_n}(A-x) \leq \mathbf{1}_{A_1}(x) \mathbf{E} e^{\langle \lambda, \xi(t_1) \rangle}$, where A_1 is the set of $x \in \mathbb{R}^d$ such that $(x, y) \in A$ for some $y \in \mathbb{R}^{d(n-1)}$,

$$\int_{E_k} \mu_{t_1, \dots, t_n}(A-x) e^{-\langle \lambda, x \rangle} \tilde{Q}_k(d\lambda) \leq c \mathbf{1}_{A_1}(x) e^{-\langle \lambda, x \rangle},$$

where c is the supremum of $\mathbf{E}e^{\langle \lambda, \xi(t_1) \rangle}$ for λ from the closure of E_1 . This supremum is finite, since $\mathbf{E}e^{\langle \lambda, \xi(t) \rangle}$ is analytic, hence continuous, in its domain U . Since A_1 is bounded,

$$\int_{\mathbb{R}^d} c \mathbf{1}_{A_1}(x) e^{-\langle \lambda, x \rangle} < \infty$$

and the Lebesgue dominated convergence theorem yields

$$\begin{aligned} \lim_{k \rightarrow \infty} \tilde{\Lambda}_k * P_{t_1, \dots, t_n}(A) &= \int_{\mathbb{R}^d} \lim_{k \rightarrow \infty} \int_{E_k} \mu_{t_1, \dots, t_n}(A - x) e^{-\langle \lambda, x \rangle} \tilde{Q}_k(d\lambda) dx \\ &= \int_{\mathbb{R}^d} \mu_{t_1, \dots, t_n}(A - x) e^{-\langle \lambda_0, x \rangle} dx = \mathbf{e}_{\lambda_0} * P_{t_1, \dots, t_n}(A), \end{aligned}$$

where the second equality follows by a similar argument as (4.10). \square

Thus, if Λ is a mixture of exponential measures, then the mixing measure Q is supported by the set $\cap_{t_1, t_2 \in \mathbb{R}} E_{P_{t_1} P_{t_2}}$.

Proposition 4.8. *Let $\lambda_1, \lambda_2 \in \mathbb{R}^d$ with $\lambda_1 \neq \lambda_2$. If the Gaussian systems $(\mathbf{e}_{\lambda_1}, \xi)$ and $(\mathbf{e}_{\lambda_2}, \xi)$ are stationary, then the one-dimensional stochastic process $\langle \xi - \mathbf{E}\xi, \lambda_2 - \lambda_1 \rangle$ is stationary.*

Proof. Writing (4.6) for λ_i , $i = 1, 2$, we arrive at

$$W(t) - \frac{1}{2} \Sigma(t, t) \lambda_1 + b_1(t) + c_1 = W(t) - \frac{1}{2} \Sigma(t, t) \lambda_2 + b_2(t) + c_2.$$

Since $\Sigma(t_2, t_1) - \Sigma(t_1, t_2)$ is a skew symmetric matrix, (4.7) implies that

$$\langle \lambda_2, b_1(t_2) - b_1(t_1) \rangle + \langle \lambda_1, b_2(t_2) - b_2(t_1) \rangle \quad (4.12)$$

is invariant after (t_1, t_2) is replaced by $(t_1 + s, t_2 + s)$. Denote shortly $\Delta\lambda = \lambda_1 - \lambda_2$. Rewriting (4.12) yields

$$\begin{aligned} \langle \Delta\lambda, \Sigma(t_1, t_1) \Delta\lambda \rangle - \langle \Delta\lambda, \Sigma(t_2, t_2) \Delta\lambda \rangle \\ = \langle \Delta\lambda, \Sigma(t_1 + s, t_1 + s) \Delta\lambda \rangle - \langle \Delta\lambda, \Sigma(t_2 + s, t_2 + s) \Delta\lambda \rangle. \end{aligned}$$

By [5, Lemma 2], the function $\langle \Delta\lambda, \Sigma(t, t) \Delta\lambda \rangle$ is an additive function plus a constant. In view of the positive definiteness of $\Sigma(t, t)$, we conclude that $\langle \Delta\lambda, \Sigma(t, t) \Delta\lambda \rangle$ is constant for all t . The statement follows from the fact that a Gaussian process with stationary increments and constant variance is itself stationary. \square

The following result characterises stationary particle systems in case the two-sided Déný equation reduces to the one-sided one.

Corollary 4.9. *Assume that ξ and Λ generate a stationary particle system, where ξ is a Gaussian process such that $P_{t_1} = P_{t_2} * \sigma$ for a Gaussian measure σ and some $t_1 \neq t_2$. If no linear combination of the components of $\xi - \mathbf{E}\xi$ is stationary, then $\Lambda = c\mathfrak{e}_\lambda$ for some $c > 0$ and ξ is given by (4.6).*

Proof. The Déný theorem implies that Λ is a mixture of exponential measures, so the result follows from Proposition 4.8. \square

In particular, Corollary 4.9 applies if $\xi(t)$ is a.s. deterministic for at least one t , for instance if $\xi(0) = 0$. Furthermore, it yields the result of [5] for non-stationary univariate process ξ .

Example 4.10. Let $\xi^1(t) = \xi^2(t) = W(t) - at/2$, where W is the standard Brownian motion and $a \in \mathbb{R}$. Then $\Lambda = \int_{\mathbb{R}} \mathfrak{e}_{(a+\lambda, -\lambda)} Q(d\lambda)$ for a measure Q on \mathbb{R} satisfying the integrability condition and $\xi = (\xi^1, \xi^2)$ generate a stationary particle system.

4.3 Measures with exponential polynomial densities.

Assume that Λ has the density

$$p(x)e^{-\langle \lambda, x \rangle} = \sum_{|\alpha| \leq k} c_\alpha x^\alpha e^{-\langle -\lambda, x \rangle},$$

where $p(x)$ is a non-negative polynomial of degree k . We use the multi-index notation, i.e. $\alpha = (\alpha^1, \dots, \alpha^d)$, $|\alpha| = \alpha^1 + \dots + \alpha^d$ and $x^\alpha = (x^1)^{\alpha^1} \dots (x^d)^{\alpha^d}$. Note that one can also consider solutions of convolution equations with not necessarily non-negative polynomials, which however do not admit a point processes interpretation. Nonetheless, even then we speak about stationary particle systems.

Theorem 4.11. *If the particle system $(p(x)e^{-\langle \lambda, x \rangle}, \xi)$ for a polynomial p is stationary, then the particle system $(q(x)e^{-\langle \lambda, x \rangle}, \xi)$ is stationary for each polynomial q obtained as a partial derivative of p .*

Proof. For each n , bounded Borel set A in \mathbb{R}^{dn} and $x \in \mathbb{R}^d$,

$$\begin{aligned}\Lambda_{t_1, \dots, t_n}(A+x) &= \int_{\mathbb{R}^d} P_{t_1, \dots, t_n}(A+x-z)p(z)e^{-\langle \lambda, z \rangle} dz \\ &= \int_{\mathbb{R}^d} P_{t_1, \dots, t_n}(A-u)p(u+x)e^{-\langle \lambda, u+x \rangle} du \\ &= \sum_{\beta \geq 0} \frac{1}{\beta!} x^\beta e^{-\langle \lambda, x \rangle} \int_{\mathbb{R}^d} P_{t_1, \dots, t_n}(A-u)q_\beta(u)e^{-\langle \lambda, u \rangle} du,\end{aligned}$$

where q_β is the partial derivative of p of order β and $\beta! = \beta^1! \dots \beta^d!$. The stationarity of the particle system and the uniqueness of the polynomial imply that the coefficients of the polynomial do not change, and so the statement of theorem follows. \square

Theorem 4.12. *The process ξ and Λ with density $\sum_{i=1}^n p_i(x)e^{-\langle \lambda_i, x \rangle}$ generate a stationary particle system if and only if ξ and measure with density $p_i(x)e^{-\langle \lambda_i, x \rangle}$ form stationary particle systems for all $i = 1, \dots, n$.*

Proof. As in the proof of Theorem 4.7, define E_1 and E_2 , where now both these sets are finite. Since the exponential grows faster than polynomial, for each bounded Borel A and sufficiently large h ,

$$\begin{aligned}\Lambda_1 * (P_{t_1, \dots, t_n} - P_{t_1+s, \dots, t_n+s})(A+hv) &> e^{-1} \Lambda_1 * (P_{t_1, \dots, t_n} - P_{t_1+s, \dots, t_n+s})(A), \\ \Lambda_2 * (P_{t_1+s, \dots, t_n+s} - P_{t_1, \dots, t_n})(A+hv) &\leq e^{-1} \Lambda_2 * (P_{t_1+s, \dots, t_n+s} - P_{t_1, \dots, t_n})(A),\end{aligned}$$

which eventually leads to a contradiction as in the proof of Theorem 4.7. \square

Theorem 4.13. *Assume that $\mathbf{E}e^{\langle u, \xi(t) \rangle} < \infty$ for all $t \in \mathbb{R}$ and all u from an open neighbourhood of λ . Then the process ξ and the measure with exponential polynomial density $p(x)e^{-\langle \lambda, x \rangle}$ generate a stationary particle system if and only if*

$$\begin{aligned}q\left(\frac{\partial}{\partial x}\right)\varphi_{t_1, \dots, t_n}(-ix - \sum_{i=2}^n u_i, u_2, \dots, u_n)|_{x=\lambda} \\ = q\left(\frac{\partial}{\partial x}\right)\varphi_{t_1+s, \dots, t_n+s}(-ix - \sum_{i=2}^n u_i, u_2, \dots, u_n)|_{x=\lambda}, \quad (4.13)\end{aligned}$$

for all partial derivatives q of p , all $n \geq 1$, $s, t_1, \dots, t_n \in \mathbb{R}$ and $u_1, \dots, u_n \in \mathbb{R}^d$ with $\sum_{i=1}^n u_i = 0$.

Proof. Denote shortly $\Delta\xi = (\xi(t_2) - \xi(t_1), \dots, \xi(t_n) - \xi(t_1))$. Similarly to (4.3), for a bounded Borel A ,

$$\begin{aligned}\Lambda_{t_1, \dots, t_n}(A) &= \int_{\mathbb{R}^d} \mathbf{E} \left[\mathbf{1}_{A-z}(0, \Delta\xi) e^{\langle \lambda, \xi(t_1) \rangle} p(z - \xi(t_1)) \right] e^{-\langle \lambda, z \rangle} dz \\ &= \sum_{\beta \geq 0} \frac{1}{\beta!} (-1)^{|\beta|} \int_{\mathbb{R}^d} \mathbf{E} \left[\mathbf{1}_{A-z}(0, \Delta\xi) \xi(t_1)^\beta e^{\langle x, \xi(t_1) \rangle} \right]_{x=\lambda} q_\beta(z) e^{-\langle \lambda, z \rangle} dz \\ &= \sum_{\beta \geq 0} \frac{1}{\beta!} (-1)^{|\beta|} \int_{\mathbb{R}^d} \frac{\partial^{|\beta|}}{\partial x^\beta} \mu_{t_1, \dots, t_n}(A - z) |_{x=\lambda} q_\beta(z) e^{-\langle \lambda, z \rangle} dz,\end{aligned}$$

where q_β denotes the β 'th partial derivative of p . By Theorems 4.1 and 4.11 the value of $\Lambda_{t_1, \dots, t_n}(A)$ is invariant for time shifts if and only if all the partial derivatives of μ are invariant. Taking the Fourier transform yields the claim. \square

Now assume that ξ is Gaussian. If ξ and Λ with density $p(x)e^{-\langle \lambda, x \rangle}$ generate a stationary particle system, then ξ and \mathfrak{e}_λ also do, so that ξ is described by Theorem 4.4.

Example 4.14. While in the univariate case Gaussian systems with a positive exponential polynomial density do not exist unless the polynomial part is constant, the convolution equation can be satisfied with a signed measure Λ . For instance, the one-dimensional signed measures with density x^{2k+1} , $k \geq 1$, together with the standard Brownian motion form stationary particle systems.

4.4 Exponential measures on subspaces

Now assume that Λ is supported by a linear subspace \mathbb{H} of \mathbb{R}^d . Denote by $\mathfrak{e}_\lambda^\mathbb{H}$ the measure on \mathbb{H} with density $e^{-\langle \lambda, x \rangle}$, $x \in \mathbb{H}$. The corresponding Poisson point process is then a subset of \mathbb{H} . Without loss of generality, it is possible to assume that $\lambda \in \mathbb{H}$ and otherwise consider its orthogonal projection on \mathbb{H} , which results in the same density. Denote by $\xi^\mathbb{H}(t)$ the orthogonal projection of $\xi(t)$ onto \mathbb{H} and let $\xi^\perp = \xi - \xi^\mathbb{H}$.

Theorem 4.15. *Assume that (4.1) holds. The process ξ and $\mathfrak{e}_\lambda^\mathbb{H}$ generate a stationary particle system if and only if (4.2) holds for all $n \geq 1$, $s, t_1, \dots, t_n \in \mathbb{R}$ and $u_1, \dots, u_n \in \mathbb{R}^d$ such that $\sum_{i=1}^n u_i$ is orthogonal to \mathbb{H} .*

Proof. The proof is similar to the proof of Theorem 4.1 with

$$\mu(A) = \mathbf{E} \left[\mathbf{1}_{(\xi(t_1) - \xi^\mathbb{H}(t_1), \dots, \xi(t_n) - \xi^\mathbb{H}(t_n)) \in A} e^{\langle \lambda, \xi^\mathbb{H}(t_1) \rangle} \right].$$

□

The following theorem concerns the Gaussian case. Let $m^{\mathbb{H}}, m^{\perp}$ be the expectations of $\xi^{\mathbb{H}}, \xi^{\perp}$. Furthermore, let $\Sigma^{\mathbb{H}}(t_1, t_2)$ (respectively $\Sigma^{\perp}(t_1, t_2)$ and $C(t_1, t_2)$) be the covariance matrix of $\xi^{\mathbb{H}}(t_1)$ and $\xi^{\mathbb{H}}(t_2)$ (respectively of $\xi^{\perp}(t_1)$ and $\xi^{\perp}(t_2)$ and of $\xi^{\mathbb{H}}(t_1)$ and $\xi^{\perp}(t_2)$). Finally, $\Gamma^{\mathbb{H}}(t_1, t_2)$ denotes the variogram of $\xi^{\mathbb{H}}$.

Theorem 4.16. *A Gaussian stochastic process ξ and measure $\mathfrak{e}_{\lambda}^{\mathbb{H}}$ generate a stationary particle system if and only if ξ satisfies the following conditions.*

- (i) $\xi^{\mathbb{H}}$ has representation (4.6) described in Theorem 4.4.
- (ii) $\xi^{\perp} - m^{\perp}$ is stationary.
- (iii) $C(t_1 + s, t_2 + s) = C(t_1, t_2) + B(s)$ for all $s, t_1, t_2 \in \mathbb{R}$, where $B(s)$ is a matrix valued additive function.
- (iv) $m^{\perp}(t) = m^{\perp}(0) - B(t)^{\top} \lambda$ for all $t \in \mathbb{R}$.

Proof. By applying a linear transformation, it is easy to reduce the situation to the case of Λ supported by the plane \mathbb{H} spanned by the first $k < d$ basis vectors in \mathbb{R}^d . If $\xi(t) = (\xi^1(t), \dots, \xi^d(t))$, then $\xi^{\mathbb{H}} = (\xi^1(t), \dots, \xi^k(t), 0, \dots, 0)$ and $\xi^{\perp} = (0, \dots, 0, \xi^{(k+1)}(t), \dots, \xi^d(t))$. By Theorem 4.15, consider the Laplace transforms with $\lambda - \sum u_i$ being zeroes in its first k coordinates. As in Theorem 4.4, consider the space \mathbb{L} that contains (u_1, u_2) with $u_1^i + u_2^i = 0$ for $i = 1, \dots, k$ and $a = (\lambda, 0)$. Then

$$A = \begin{pmatrix} I & -I_k \\ -I_k & I \end{pmatrix}$$

is the projection on \mathbb{L} , where I_k is the matrix with first k diagonal entries being one and otherwise zeroes. Then

$$A^{\top} \Sigma A = \begin{pmatrix} \Gamma_{12}^{\mathbb{H}} & C_{11} - C_{21} & -\Gamma_{12}^{\mathbb{H}} & C_{12} - C_{22} \\ C_{11}^{\top} - C_{12}^{\top} & \Sigma_{11}^{\perp} & C_{12}^{\top} - C_{11}^{\top} & \Sigma_{12}^{\perp} \\ -\Gamma_{12}^{\mathbb{H}} & C_{21} - C_{11} & \Gamma_{12}^{\mathbb{H}} & C_{22} - C_{12} \\ C_{21}^{\top} - C_{22}^{\top} & \Sigma_{21}^{\perp} & C_{22}^{\top} - C_{21}^{\top} & \Sigma_{22}^{\perp} \end{pmatrix}, \quad (4.14)$$

$$A^{\top}(m + \Sigma a) = \begin{pmatrix} m_1^{\mathbb{H}} + \Sigma_{11}^{\mathbb{H}} \lambda - m_2^{\mathbb{H}} - \Sigma_{21}^{\mathbb{H}} \lambda \\ m_1^{\perp} + C_{11}^{\top} \lambda \\ m_2^{\mathbb{H}} + \Sigma_{21}^{\mathbb{H}} \lambda - m_1^{\mathbb{H}} - \Sigma_{11}^{\mathbb{H}} \lambda \\ m_2^{\perp} + C_{21}^{\top} \lambda \end{pmatrix}, \quad (4.15)$$

$$\langle m, a \rangle + \frac{1}{2} \langle a, \Sigma a \rangle = \langle m_1^{\mathbb{H}}, \lambda \rangle + \frac{1}{2} \langle \lambda, \Sigma_{11}^{\mathbb{H}} \lambda \rangle, \quad (4.16)$$

where $\Sigma_{ij} = \Sigma(t_i, t_j)$, $\Gamma_{ij} = \Gamma(t_i, t_j)$ and $C_{ij} = C(t_i, t_j)$. The invariance of $\Gamma_{ij}^{\mathbb{H}}$, the first row of (4.15) and (4.16) imply the representation of $\xi^{\mathbb{H}}$. The invariance of Σ_{ij}^{\perp} in (4.14) yields the stationarity of $\xi^{\perp} - m^{\perp}$.

Define $B_{t_1, t_2}(s) = C(t_1 + s, t_2 + s) - C(t_1, t_2)$. The invariance of $C(t_1, t_2) - C(t_1, t_1)$ and $C(t_2, t_1) - C(t_1, t_1)$ with respect to translation of the time arguments yields that $B_{t_1, t_2}(s_1 + s_2) = B_{t_1, t_2}(s_1) + B_{t_1, t_2}(s_2)$ and that $B_{t_1, t_2}(s)$ does not depend on t_1, t_2 . For every matrix valued additive function B the covariances of the form $C(t_1 + s, t_2 + s) = C(t_1, t_2) + B(s)$ satisfy the invariance properties. Finally (iv) is obtained by considering the second and fourth row of (4.15).

For the sufficiency note that the Laplace transform of the random vector $(\xi(t_1), \dots, \xi(t_n))$ at the point

$$\left(\left(\lambda - \sum_{i=2}^n u_i^{\mathbb{H}} \right), \begin{pmatrix} u_2^{\mathbb{H}} \\ u_2^{\perp} \end{pmatrix}, \dots, \begin{pmatrix} u_n^{\mathbb{H}} \\ u_n^{\perp} \end{pmatrix} \right)$$

consists of a combination of similar elements as given by (4.14), (4.15) and (4.16), where $u_i^{\mathbb{H}}$ and u_i^{\perp} denote orthogonal projection of u_i on \mathbb{H} and its complement. \square

The following example shows that there exists stationary systems generated by a process where ξ^{\perp} is not stationary.

Example 4.17. Let $\mathbb{H} = \mathbb{R} \times \{0\} \subset \mathbb{R}^2$ and consider $\xi = (\xi^1, \xi^2)$ with $\xi^1(t) = Zt - \frac{1}{2}\lambda t^2$ and $\xi^2(t) = Z - \lambda t$ for the standard Gaussian variable Z and $\lambda \in \mathbb{R}$. By Theorem 4.16, ξ together with $\mathfrak{e}_{(\lambda, 0)}^{\mathbb{H}}$ form a stationary particle system.

5 Multivariate Brown–Resnick processes

Consider a special case of the particle system that appears if the Poisson process Π lives on the diagonal line $\mathbb{H} = \{x^1 = \dots = x^d\}$ in \mathbb{R}^d . In this case, instead of the additive particle system it is convenient to consider the *multiplicative* particle system

$$N^e(t) = \{y_i e^{\xi_i(t)}, i \geq 1\}, \quad t \in \mathbb{R}, \quad (5.1)$$

where $\{y_i : i \geq 1\} = \Pi^e$ is a Poisson process on $(0, \infty)$ with intensity measure Λ^e and independent of $\{\xi_n, n \geq 1\}$, which are i.i.d. copies of a \mathbb{R}^d -valued stochastic process $\xi(t)$ satisfying

$$\mathbf{E} e^{\xi(t)} < \infty \quad \text{for all } t \in \mathbb{R}. \quad (5.2)$$

Note that the exponential is applied coordinatewisely and the finiteness of expectation means that all its coordinates are finite. Then the intensity measure of $N^e(t_1, \dots, t_n)$ is locally finite and given by

$$\Lambda_{t_1, \dots, t_n}^e(A) = \int_{(0, \infty)} \mathbf{P} \left\{ (e^{\xi(t_1)}, \dots, e^{\xi(t_n)}) \in y^{-1}A \right\} \Lambda^e(dy) \quad (5.3)$$

for all Borel $A \subset \mathbb{R}^{dn}$.

Assume that Π^e has intensity measure $\Lambda^e(dy) = y^{-2}dy$ and define a process η with values in \mathbb{R}^d by

$$\eta(t) = \bigvee_{i=1}^{\infty} y_i e^{\xi_i(t)}, \quad t \in \mathbb{R}, \quad (5.4)$$

where the maximum is taken coordinatewisely. It is well known that the process η is max-stable with unit Fréchet margins, see [4]. In order to determine the finite-dimensional distributions of η note that the event $\{\eta(t_1) \leq z_1, \dots, \eta(t_n) \leq z_n\}$ (with coordinatewise inequalities) is equivalent to the fact that no point of the process N^e defined by (5.1) lies outside $A = (0, z_1] \times \dots \times (0, z_n]$. The latter probability equals $\exp\{-\Lambda_{t_1, \dots, t_n}^e((0, \infty)^{dn} \setminus A)\}$, so that (5.3) yields that

$$\mathbf{P} \{\eta(t_1) \leq z_1, \dots, \eta(t_n) \leq z_n\} = \exp \left\{ -\mathbf{E} \max_{j,k} \left(e^{\xi^k(t_j)} / z_j^k \right) \right\} \quad (5.5)$$

for all $t_1, \dots, t_n \in \mathbb{R}$ and $z_1, \dots, z_n \in \mathbb{R}^d$. Applying this for $n = 1$, it is easily seen that condition (5.2) ensures that $\eta(t)$ is a.s. finite for every $t \in \mathbb{R}$. Furthermore, the above argument shows that the finite-dimensional distributions of η uniquely determine the finite-dimensional distributions of N^e . In particular, N^e is stationary if and only if η is stationary. The following definition appears in [6], however only for stochastic processes with values in the real line.

Definition 5.1 (see [6]). A stochastic process ξ satisfying (5.2) is called *Brown–Resnick stationary* if the process η defined by (5.4) is stationary.

Theorem 5.2. A stochastic process $\xi(t)$, $t \in \mathbb{R}$, is *Brown–Resnick stationary* if and only if

$$\varphi_{t_1, \dots, t_n}(u_1 - id^{-1}\mathbf{1}, u_2, \dots, u_n) = \varphi_{t_1+s, \dots, t_n+s}(u_1 - id^{-1}\mathbf{1}, u_2, \dots, u_n)$$

for all $s, t_1, \dots, t_n \in \mathbb{R}$ and $u_1, \dots, u_n \in \mathbb{R}^d$ satisfying $\sum_{i,j=1}^n u_i^j = 0$, where $\mathbf{1}$ is the vector with all components equal to one.

Proof. If we consider the additive system, i.e. let $x_i = \log(y_i)$ and $N(t) = \{x_i + \xi_i(t), i \geq 1\}$, then the Brown–Resnick construction corresponds to the situation where $\{x_i, i \geq 1\}$ is a Poisson process on the line $\mathbb{H} = \{(x, \dots, x) : x \in \mathbb{R}\}$ in \mathbb{R}^d with intensity e^{-x} , $x \in \mathbb{R}$. Then the result follows from Theorem 4.15. \square

Since the measure Λ^e is prescribed, the Brown–Resnick stationary processes form a subclass of stationary particle systems with special (namely diagonal) intensity measures supported by the line \mathbb{H} . Then $\xi^{\mathbb{H}}$ is the vector with all components being $\bar{\xi} = d^{-1} \sum_{i=1}^d \xi^i$ and $\xi^\perp = \xi - \xi^{\mathbb{H}}$. The following result characterises all pairs of a Gaussian process and a Poisson process on \mathbb{H} that yield stationary particle systems.

Theorem 5.3. *A Gaussian process ξ , such that $\langle v, \xi \rangle$ is not stationary for some $v \notin \mathbb{H}$, and a locally finite measure Λ on the line \mathbb{H} generate a stationary particle system if and only if Λ is proportional to $\mathfrak{e}_\lambda^{\mathbb{H}}$,*

$$\bar{\xi}(t) = \begin{cases} W(t) + b(t) + c & \text{if } \lambda = 0, \\ W(t) - \frac{1}{2}\lambda\sigma^2(t) + c & \text{if } \lambda \neq 0, \end{cases}$$

where $W(t)$, $t \in \mathbb{R}$, is a centred univariate Gaussian process with stationary increments and variance $\sigma^2(t)$, b is an additive univariate function, $c \in \mathbb{R}$ is a constant, and $\xi^{\mathbb{H}}, \xi^\perp$ satisfy the conditions (ii)-(iv) of Theorem 4.16.

Proof. The sufficiency is easy to show. For the necessity, note that since Λ is concentrated on \mathbb{H} , the projected particle system $N_v = \{\langle v, x_i + \xi_i \rangle, i \geq 1\}$ is also a particle system generated by a non-stationary Gaussian process. The characterisation of univariate particle systems from [5] yields that the projected Λ is an exponential measure, and so is Λ itself, since it is supported by one-dimensional subspace. The rest of the proof follows from Theorem 4.16. \square

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